

SMALL DATA SCATTERING FOR ENERGY-SUBCRITICAL AND CRITICAL NONLINEAR KLEIN GORDON EQUATIONS ON PRODUCT SPACES

LYSIANNE HARI AND NICOLA VISCIGLIA

ABSTRACT. We study small data scattering of solutions to Nonlinear Klein-Gordon equations with suitable pure power nonlinearities, posed on $\mathbf{R}^d \times \mathcal{M}^k$ with $k \leq 2$ and $d \geq 1$ and \mathcal{M}^k a compact Riemannian manifold. As a special case we cover the H^1 -critical NLKG on $\mathbf{R} \times M^2$.

Keywords: Nonlinear Klein-Gordon equation on manifolds, Scattering, Strichartz estimates.

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1. INTRODUCTION

Let $d \geq 1$ and $k = 1, 2$, such that $3 \leq d + k \leq 6$. We consider the Cauchy problem for the Nonlinear Klein-Gordon equation (NLKG) posed on $R^d \times \mathcal{M}^k$

$$(1.1) \quad \partial_t^2 u - \Delta_{x,y} u + u = \pm |u|^{p-1} u, \quad u|_{t=0}(x, y) = f(x, y), \quad \partial_t u|_{t=0}(x, y) = g(x, y),$$

with $(t, x, y) \in \mathbf{R}_t \times \mathbf{R}_x^d \times \mathcal{M}_y^k$, where

- ★ $d \geq 1$,
- ★ \mathcal{M}_y^k is a compact Riemannian manifold of dimension k ,
- ★ $\Delta_{x,y} = \Delta_x + \Delta_y$, where $\Delta_x = \sum_{j=1}^d \partial_{x_j}^2$ is the Laplace operator associated to the flat metric on \mathbf{R}^d and Δ_y is the Laplace-Beltrami operator on \mathcal{M}_y^k . We write dy the volume element of \mathcal{M}_y^k , and

$$Vol(\mathcal{M}^k) = \int_{\mathcal{M}^k} dy$$

the (finite) volume of \mathcal{M}^k ,

- ★ p is such that $p_0 \leq p \leq p_c$ where p_c is the H^1 -critical exponent for the whole dimension $d + k$:

$$(1.2) \quad p_c = 1 + \frac{4}{d + k - 2},$$

and where

$$(1.3) \quad p_0 = \max \left(2, 1 + \frac{4}{d} \right)$$

is larger than the L^2 -critical exponent on \mathbf{R}^d .

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The aim of this paper is to investigate the persistence of scattering properties for the nonlinear Klein-Gordon equation (NLKG), considered on product spaces of total dimension larger than three. The study of dispersive PDE's posed on product spaces was first initiated for the Schrödinger equation (NLS) (see [21, 53] for problems involving global well-posedness, [19, 20, 54, 60, 61] for long time asymptotics and [55] for studies about ground states). On one hand, choosing carefully the parameters, one has global existence and scattering results on Euclidean spaces \mathbf{R}^d , in particular for small data. On the other hand, considering the equation on compact Riemannian manifolds, the previous assertion fails. A question then arises when the equation is posed on a product of Euclidean space and compact Riemannian manifold: is the dispersive nature of the Euclidean part sufficient to prevail and rule the behaviour of the whole solution at infinity ?

The same question comes up for other dispersive PDEs in such settings; we focus on the nonlinear Klein-Gordon equation in this paper. A particular case is when \mathcal{M}^k is \mathbb{T}^k the flat torus of dimension k : this kind of semiperiodic settings is involved in wave guide theories, especially in the case $d + k = 3$. Besides, there are numerous references about the cubic (NLKG) or (NLW) posed on specific manifolds in general relativity. However, we will see that the topological structure of the manifold is not relevant in our study and that the small data theory does not need any specific property on \mathcal{M}^k .

The analysis of existence and scattering properties on \mathbf{R}^d can be found in several references. It is well-known that scattering for small H^1 data holds for

$$1 + \frac{4}{d} \leq p \leq 1 + \frac{4}{d-2}$$

since existence of wave operators and asymptotic completeness is proved in those cases.

We do not make any exhaustive list of all previous works dealing with similar problems but for the particular case of small data scattering results, we refer the reader to [9, 16, 18, 43, 44, 52, 59]. All dimensions and all p lying between the L^2 -critical and the H^1 -critical exponents are not necessarily handled in these references.

Among several papers dealing with small data scattering in \mathbf{R}^d for (NLW) (case $m^2 = 0$), we give [28, 35, 31, 33, 32, 45, 50, 51] (some of the references also contain the case $m^2 > 0$) and references therein.

More recent results can be found in [34, 62, 63], and similar problems in different frameworks are discussed in [35], where the nonlinearity is a sum of nonlinear terms with exponents from L^2 -critical to H^s -critical nonlinearities with $s < d/2$, in [46, 47] for (NLKG) with nonlocal nonlinearities; [27] for variable coefficients added in the cubic nonlinearity.

For a deeper discussion on well-posedness and large data scattering on \mathbf{R}^d , for energy subcritical and critical nonlinearities, (for both defocusing and focusing cases) see [22, 26, 36, 37, 38, 39, 40]. More recent small data results - such as low dimensions results - can be deduced from the references above. We do not comment the large data problem since it will be handled in an ongoing work.

Existence of solutions for Nonlinear Klein-Gordon posed on tori and spheres have been studied, mainly in papers of Delort [11, 12], Delort-Szeftel [13, 14], Fang-Zhang [15].

There are also several results dealing with (NLKG) in various settings (with potentials and/or in other type of space structures). Several results about the decay of solutions and scattering for the cubic (NLW) and (NLKG) posed on Schwarzschild manifolds can be related to our framework. In fact, the equation is posed on a related static spacetime with product structure

$$\mathbf{R}_t \times (2M, \infty) \times S^2.$$

But the spacelike hypersurface is equipped with a product metric of the form $g = dx^2 + c^2(x)g_0$, where g_0 is a metric on the compact manifold; in our case $c^2 \equiv 1$. However under some (strong) assumptions, one could reduce the study to (NLKG) similar to our problem but with additionnal terms that seem to be difficult to handle. We give a non-exhaustive list of such references (for a deeper discussion of such frameworks involved in general relativity, we refer the reader to references given there): Blue-Soffer [6, 7] and handle decay estimates for (NLW), whereas existence and asymptotical studies for (NLKG) are performed by Bachelot, Nicolas [2, 41], [42] (on Kerr metrics). We do not mention works about more general decay estimates, or works about the linear wave equation or wave maps posed on manifolds.

Since the Nonlinear Dirac equation (NLD) is intimately linked with (NLKG), results about it should be added to complete the references. Problems involving scattering for (NLD) are discussed for example in [5, 3, 48, 49] (see also [4] for systems of (NLD)-(NLKG)).

The main tools to prove scattering are Strichartz estimates, whose proof will be detailed in Section 2.2. We will use the knowledge on the flat part to deduce Strichartz estimates on the whole product space, as it is performed in [60].

Remark 1.1 (Restrictions on p). Let us give details on the restrictions made on p . Those restrictions imply ones made on the dimensions d, k at the beginning.

First, we need $p \geq 2$. In fact, smaller p cannot be handled with our estimates. We would need more general Strichartz estimates that are not available with our argument, detailed in Section 2.2. The homogeneous term will be estimated in $L_t^1 L_{x,y}^2$ norms. By considering $p \geq 2$, we only deal with $p_c \geq 2$, giving $d + k \leq 6$. We also see that $1 + 4/d \leq 1 + 4/(d + k - 2)$ which yields $k \leq 2$.

As for (NLS), it is quite natural to restrict $p \geq p_0$. In fact, considering data which are constant in their y -variables, it is easy to see that for $p < 1 + 4/d$, the analysis is reduced to L^2 -subcritical case on \mathbf{R}^d , for which no scattering in energy space seems to be available.

Besides as in [60], one can prove scattering for some H^1 -supercritical cases $p \geq p_c$ in anisotropic Sobolev spaces of higher regularity (see Theorem 1.4).

1.1. Notations. We introduce some notations and definitions that will be useful in the paper.

Notation. For any Lebesgue exponent $q \geq 1$, we write q' its dual:

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

We define admissibility for Schrödinger and Klein-Gordon:

Definition 1.2. A pair (q, r) is **admissible** if $2 \leq r \leq \frac{2d}{d-2}$ ($2 \leq r \leq \infty$ if $d = 1$, $2 \leq r < \infty$ if $d = 2$) and

$$\frac{2}{q} = d \left(\frac{1}{2} - \frac{1}{r} \right).$$

We denote \mathcal{F} the Fourier transform with respect to the variable on \mathbf{R}_x^d and \mathcal{F}^{-1} the inverse transform. With the following partitions of the unity

$$1 = \chi_0(\xi) + \sum_{j>0} \varphi_j(\xi), \forall \xi,$$

where

- ★ χ_0 the cut-off function satisfying $\chi_0(\xi) = 1$ for $|\xi| < 1$ and $\chi_0(\xi) = 0$ for $|\xi| > 2$,
- ★ $\varphi_j(\xi) = \chi_0(2^{-j}\xi) - \chi_0(2^{-j+1}\xi)$,

we define the following operators

$$\begin{aligned} P_0 f &:= \mathcal{F}^{-1} \chi_0 \mathcal{F}(f), \\ P_j f &:= \mathcal{F}^{-1} \varphi_j \mathcal{F}(f), \quad \forall j > 0, \end{aligned}$$

We denote by $\mathcal{S}'(\mathbf{R}^d)$ the set of all tempered distributions on \mathbf{R}^d .

Notation (Besov spaces on Euclidean spaces). Let $-\infty < s < \infty$, $0 < q, r \leq \infty$. Then, for $0 < q \leq \infty$, the Besov space $B_{q,r}^s$ is defined by

$$B_{q,r}^s(\mathbf{R}^d) = \left\{ f \in \mathcal{S}'(\mathbf{R}^d) \mid \left(\sum_{j=0}^{\infty} 2^{jsr} \|P_j f\|_{L^q(\mathbf{R}^d)}^r \right)^{1/r} < \infty \right\}$$

We denote by $C_0^\infty(\mathbb{M})$ the set of test functions on the manifold \mathbb{M} , which can be \mathbf{R}^d , \mathcal{M}^k or $\mathbf{R}^d \times \mathcal{M}^k$.

Notation (Sobolev spaces on Euclidean spaces, compact manifolds and product spaces). Let $d \geq 1$. For any $s \geq 0$ and $1 \leq q \leq \infty$, we write

- ★ (Inhomogeneous Sobolev spaces) $W^{s,q}(\mathbf{R}^d)$ the completion of $C_0^\infty(\mathbf{R}^d)$ with respect to

$$\left\| \mathcal{F}^{-1}((1 + |\xi|^2)^{s/2} \mathcal{F}(f)) \right\|_{L^q(\mathbf{R}^d)}.$$

- ★ (Homogeneous Sobolev spaces) $\dot{W}^{s,q}(\mathbf{R}^d)$ the completion of $C_0^\infty(\mathbf{R}^d)$ with respect to

$$\left\| \mathcal{F}^{-1}(|\xi|^s \mathcal{F}(f)) \right\|_{L^q(\mathbf{R}^d)}.$$

When $q = 2$, we will write $W^{s,2}(\mathbf{R}^d) = H^s(\mathbf{R}^d)$.

For Sobolev spaces with integer derivatives $s \in \mathbf{N}$ in the x -variables, an equivalent norm involving all derivatives of order smaller than s could be used.

Let $k \geq 1$. We write $\{\lambda_j\}$ the eigenvalues of $-\Delta_y$, sorted in ascending order and taking in account the multiplicity, $\{\Phi_j(y)\}$, the eigenfunctions associated with λ_j , that are

$$(1.4) \quad -\Delta_y \Phi_j = \lambda_j \Phi_j, \quad \lambda_j \geq 0, \quad \forall j,$$

then one has an orthonormal basis of $L^2(\mathcal{M}_y^k)$ given by (1.4).

We introduce for any $q \in \mathbf{N} \setminus \{0\}$,

$$l_j^q = \left\{ \{f_j\}_{j \geq 0} \mid \left(\sum_{j \geq 0} |f_j|^q \right)^{1/q} < \infty \right\}.$$

We write

$$f(y) = \sum_{j \geq 0} f_j \Phi_j(y)$$

the decomposition of any function $f : \mathcal{M}_y^k \rightarrow \mathbf{C}$. For any $s \geq 0$ and $1 \leq q \leq \infty$, we define

★ (Inhomogeneous Sobolev spaces) $W^{s,q}(\mathcal{M}^k)$ the completion of $C_0^\infty(\mathcal{M}^k)$ with respect to

$$\left\| (1 + |\lambda_j|)^{s/2} f_j \right\|_{L_j^q}.$$

★ (Homogeneous Sobolev spaces) $\dot{W}^{s,q}(\mathcal{M}^k)$ the completion of $C_0^\infty(\mathcal{M}^k)$ with respect to

$$\left\| |\lambda_j|^{s/2} f_j \right\|_{L_j^q}.$$

When $q = 2$, we will write $W^{s,2}(\mathcal{M}^k) = H^s(\mathcal{M}^k)$.

Finally, for our product spaces $\mathbb{M} = \mathbf{R}_x^d \times \mathcal{M}_y^k$ we will use the following notations in the specific case $q = 2$. We consider λ_j defined in (1.4).

★ $H_x^\theta H_y^\rho = (1 - \Delta_x)^{-\theta/2} (1 - \Delta_y)^{-\rho/2} L_{x,y}^2$, endowed with the natural norm.

★ $\dot{H}_x^\theta \dot{H}_y^\rho = (-\Delta_x)^{-\theta/2} (-\Delta_y)^{-\rho/2} L_{x,y}^2$, endowed with the natural norms.

★ $H_{x,y}^\theta = (1 - \Delta_{x,y})^{-\theta/2} L_{x,y}^2$, endowed with the natural norm.

1.2. The results. Let us write for any $(f, g) \in H_{x,y}^1 \times L_{x,y}^2$

$$(1.5) \quad S(t)(f, g) = \cos\left(t \sqrt{1 - \Delta_{x,y}}\right) f + \frac{\sin\left(t \sqrt{1 - \Delta_{x,y}}\right)}{\sqrt{1 - \Delta_{x,y}}} g$$

$$(1.6) \quad \overrightarrow{S(t)}(f, g) = (S(t)(f, g), \partial_t S(t)(f, g)).$$

Then we prove

Theorem 1.3. *Let $1 \leq d \leq 5$ and $k = 1, 2$ such that $3 \leq d + k \leq 6$. Consider $p_0 \leq p \leq p_c$ given by (1.2)-(1.3). Then, there exists $\delta > 0$ such that the Cauchy problem (1.1) has a unique global solution*

$$u(t, x, y) \in C^0(\mathbf{R}, H_{x,y}^1) \cap C^1(\mathbf{R}, L_{x,y}^2), \quad u(t, x, y) \in L^p(\mathbf{R}, L_{x,y}^{2p}),$$

$$(\text{ and so we have } \quad \partial_t u(t, x, y) \in C^0(\mathbf{R}, L_{x,y}^2))$$

for any initial data $(f, g) \in H_{x,y}^1 \times L_{x,y}^2$ such that $\|f\|_{H_{x,y}^1} + \|g\|_{L_{x,y}^2} < \delta$.

Moreover there exist two couples $(f^+, g^+), (f^-, g^-)$ in $H_{x,y}^1 \times L_{x,y}^2$ such that

$$\lim_{t \rightarrow \pm\infty} \left\| \overrightarrow{S(t)}(f^\pm, g^\pm) - (u(t, \cdot), \partial_t u(t, \cdot)) \right\|_{H_{x,y}^1 \times L_{x,y}^2} = 0.$$

We also give some H^1 -supercritical cases for which an additional regularity is required in the y -variable.

Theorem 1.4. *Let $1 \leq d \leq 5$, and $k \geq 1$, with $k \geq 2$ if $d = 1$. We assume $p_0 \leq p$ given by (1.3) and that*

$$p \quad \begin{cases} < \infty & \text{if } d = 1, 2, \\ \leq \frac{d^2 + 2d - 4}{d^2 - 2d} & \text{if } d \geq 3. \end{cases}$$

Then for all $\gamma > k/2$, there exists $\delta > 0$ such that the Cauchy problem (1.1) has a unique global solution

$$u(t, x, y) \in C^0(\mathbf{R}, H_{x,y}^1) \cap C^1(\mathbf{R}, L_{x,y}^2)$$

satisfying

$$(1 - \Delta_y)^{\gamma/2} u(t, x, y) \in L^p(\mathbf{R}, L_x^{2p} L_y^2),$$

for any initial data (f, g) such that $(f, g) \in (H_{x,y}^{1,\gamma} \times H_{x,y}^{0,\gamma}) \cap (H_{x,y}^1 \times L_{x,y}^2)$ and

$$\|(1 - \Delta_y)^{\gamma/2} f\|_{H_{x,y}^1} + \|(1 - \Delta_y)^{\gamma/2} g\|_{L_{x,y}^2} < \delta.$$

Moreover there exist two couples

$$(f^+, g^+), (f^-, g^-) \text{ in } (H_{x,y}^{1,\gamma} \times H_{x,y}^{0,\gamma}) \cap (H_{x,y}^1 \times L_{x,y}^2)$$

such that

$$\lim_{t \rightarrow \pm\infty} \left\| \overrightarrow{S(t)}(f^\pm, g^\pm) - (u(t, \cdot), \partial_t u(t, \cdot)) \right\|_{H_{x,y}^{1,\gamma} \times H_{x,y}^{0,\gamma}} = 0.$$

The main difference between both theorems, is that Theorem 1.4 gives scattering in higher order Sobolev spaces but for larger k and for a wider range of p . In particular, when $k = 1, 2$ we can consider some H^1 -supercritical cases

$$p_c < p, \quad \text{with } p \leq \frac{d^2 + 2d - 4}{d^2 - 2d}, \quad \text{if } 3 \leq d \leq 5,$$

whereas for $k \geq 3$, since $p_c < p_0$ we consider H^1 -supercritical cases on the whole product space that are L^2 -subcritical on \mathbf{R}^d .

Remark 1.5 (Comparison with the small data theory for (NLS)). For Theorems 1.3, one can see that no additional regularity is required. Thus, H^1 -scattering can be proved only making use of Lebesgue spaces, for some energy subcritical and critical nonlinearities; the latter being the most interesting case. As an example, consider $\mathbf{R}^3 \times \mathcal{M}^1$ and $\mathbf{R}^2 \times \mathcal{M}^2$ for which the cubic nonlinearity is $H_{x,y}^1$ -critical. In [60] one requires the smallness of the data in $H_{x_1,x_2}^0 H_{x_3,y}^{1+\varepsilon}$ or $H_{x_1,x_2}^0 H_{y_1,y_2}^{1+\varepsilon}$ and scattering is proved in those anisotropic spaces. For the mass-energy-critical exponents on $\mathbf{R}^d \times \mathcal{M}^2$, a more recent result from [54] gives H^1 -scattering assuming smallness in anisotropic spaces with $1 + \varepsilon$ derivatives in y . Small (f, g) in $H_{x,y}^1 \times L_{x,y}^2$ is enough in our case to prove H^1 -scattering. Besides, our results do not depend on the geometry of the compact manifold (more general \mathcal{M}^k are allowed whereas the explicit structure of the torus \mathbb{T}^2 was necessary in Theorem 1.1 of [19] for (NLS) posed on $\mathbf{R} \times \mathbb{T}^2$).

We also notice that for the small data theory, we do not use the finite time of propagation property.

2. ABOUT THE STRICHARTZ ESTIMATES

2.1. The strategy: from Schrödinger ([60]) to Klein-Gordon. Let us focus on more general frameworks for the Nonlinear Schrödinger equation to make a comparison. The study of well-posedness on some specific product spaces and for small data scattering were the first steps of such analysis. Some large data scattering results are also proved in specific cases (see [19, 20, 53, 54, 60, 61] and references therein).

Since our aim is to investigate the persistence of scattering properties for small data, we recall the strategy used in [60] and make a quick comparison with the difficulties arising for the Nonlinear Klein-Gordon equation. The proofs of small data scattering rely on global-in-time Strichartz

estimates, deduced from the Euclidean case and extended, in some sense, to the product spaces.

Consider

$$(2.1) \quad i\partial_t u + \Delta_{x,y} u = F, \quad u|_{t=0}(x, y) = f(x, y), \quad (t, x, y) \in \mathbf{R}_t \times \mathbf{R}_x^d \times \mathcal{M}_y^k,$$

where $d \geq 2$, $k \geq 1$. The authors of [60] are able to prove the following global in time Strichartz estimates on the whole product spaces, with a simple L^2 -norm in the y -variables

$$(2.2) \quad \|u\|_{L_t^{q_1} L_x^{r_1} L_y^2} \leq C \left[\|f\|_{L_{x,y}^2} + \|F\|_{L_t^{q'_2} L_x^{r'_2} L_y^2} \right].$$

But the anisotropic Lebesgue spaces in x and y variables bring problems, handled by considering Strichartz with derivatives.

When one wishes to use the same strategy for (1.1), the main problem appears after decomposing the functions on the basis (1.4). In fact, writing

$$u(t, x, y) = \cos\left(t \cdot \sqrt{1 - \Delta_{x,y}}\right) f(x, y) + \frac{\sin\left(t \cdot \sqrt{1 - \Delta_{x,y}}\right)}{\sqrt{1 - \Delta_{x,y}}} g(x, y) + \int_0^t \frac{\sin\left((t-s) \cdot \sqrt{1 - \Delta_{x,y}}\right)}{\sqrt{1 - \Delta_{x,y}}} F(s, x, y) ds,$$

the solution of

$$\partial_t^2 u - \Delta_{x,y} u + u = F, \quad u|_{t=0}(x, y) = f(x, y), \quad \partial_t u|_{t=0}(x, y) = g(x, y),$$

with $(t, x, y) \in \mathbf{R}_t \times \mathbf{R}_x^d \times \mathcal{M}_y^k$, one writes

$$\begin{aligned} u(t, x, y) &= \sum_j u_j(t, x) \Phi_j(y) \\ F(t, x, y) &= \sum_j F_j(t, x) \Phi_j(y) \\ f(x, y) &= \sum_j f_j(x) \Phi_j(y) \\ g(x, y) &= \sum_j g_j(x) \Phi_j(y), \end{aligned}$$

and obtains the following decoupled system of “flat” equations

$$(2.3) \quad \partial_t^2 u_j - \Delta_x u_j + u_j + \lambda_j u_j = F_j, \quad u_j|_{t=0}(x) = f_j(x), \quad \partial_t u_j|_{t=0}(x) = g_j(x).$$

It is easy to see that the Strichartz estimates involving

$$\left(\cos\left(t \cdot \sqrt{1 + \lambda_j - \Delta_x}\right), \quad \frac{\sin\left(t \cdot \sqrt{1 + \lambda_j - \Delta_x}\right)}{\sqrt{1 + \lambda_j - \Delta_x}} \right)$$

will not be independent of λ_j . Therefore the whole estimate on $\mathbf{R}^d \times \mathcal{M}_y^k$ will be altered in the y -variables for general admissible pairs.

But, unlike the Schrödinger case, we will see that in the framework of Theorem 1.3, no control of u in anisotropic Sobolev spaces will be needed: in fact, obtaining a Sobolev norm in the y -variables

is a good point. Thanks to an appropriate choice of pairs and Sobolev embeddings, we will be able to deal with the Cauchy problem (1.1) with Lebesgue spaces in y .

Remark 2.1. Let us notice that one could deal with a mass $m^2 > 0$, $m^2 \neq 1$ in (1.1)

$$\partial_t^2 u - \Delta_{x,y} u + m^2 u = \pm |u|^{p-1} u, \quad u|_{t=0}(x, y) = f(x, y), \quad \partial_t u|_{t=0}(x, y) = g(x, y).$$

The quantity m^2 should be fixed at the beginning so it would not bring any problem in all computations involved in the proofs.

Two other remarks about this mass m^2 can be made: first, even with the Nonlinear Wave equation ($m^2 = 0$), one ends up with a system of Nonlinear Klein Gordon equations (2.3) with $\lambda_j u_j$ instead of $(1 + \lambda_j) u_j$.

Then dealing with $m^2 = 0$ is not easy on arbitrary \mathcal{M}_y^k since the first eigenvalue λ_0 could be zero. Hence, the first equation of the system (2.3) would be a Wave equation, for which the Strichartz estimates are different with different admissibility. We do not deal with this case in this paper.

2.2. Strichartz estimates. We recall admissibility for the Klein-Gordon equation, which is the same as for Schrödinger in Definition 1.2: A pair (q, r) is **admissible** if $2 \leq r \leq \frac{2d}{d-2}$ ($2 \leq r \leq \infty$ if $d = 1$, $2 \leq r < \infty$ if $d = 2$) and

$$\frac{2}{q} = d \left(\frac{1}{2} - \frac{1}{r} \right).$$

We also define an exponent that will be used in the Besov spaces, in Section 2.2:

Notation. Consider (q, r) an admissible pair given by Definition 1.2. We then denote by s the following exponent:

$$(2.4) \quad s = 1 - \frac{1}{2} \left(\frac{d}{2} + 1 \right) \cdot \left(\frac{1}{r'} - \frac{1}{r} \right) = 1 - \frac{1}{2} \left(\frac{d}{2} + 1 \right) \cdot \left(1 - \frac{2}{r} \right).$$

Proposition 2.2 (Strichartz estimates). *Let $1 \leq d \leq 5$, $k \geq 1$ and assume $k \geq 2$ if $d = 1$. Consider $p_0 \leq p$ given by (1.3) and u given by*

$$u(t, x, y) = \cos \left(t \sqrt{1 - \Delta_{x,y}} \right) f(x, y) + \frac{\sin \left(t \sqrt{1 - \Delta_{x,y}} \right)}{\sqrt{1 - \Delta_{x,y}}} g(x, y) + \int_0^t \frac{\sin \left((t-s) \sqrt{1 - \Delta_{x,y}} \right)}{\sqrt{1 - \Delta_{x,y}}} F(s, x, y) ds.$$

for any f, g, F in $\mathcal{S}(\mathbf{R}^d \times \mathcal{M}^k)$. Then

(1) Assume $k \leq 2$ such that $3 \leq d + k \leq 6$ and $p_0 \leq p \leq p_c$ given by (1.2)-(1.3). Then

$$(2.5) \quad \|u\|_{L_t^p L_{x,y}^{2p}} \leq C \left[\|f\|_{H_{x,y}^1} + \|g\|_{L_{x,y}^2} + \|F\|_{L_t^1 L_{x,y}^2} \right],$$

where $C > 0$ depends on p , and might depend on $\text{Vol}(\mathcal{M}^k)$.

(2) Assume $p_0 \leq p$ given by (1.3). Assume $p \leq \frac{d^2 + 2d - 4}{d^2 - 2d}$ if $3 \leq d \leq 5$. Then

$$(2.6) \quad \|u\|_{L_t^p L_x^{2p} L_y^2} \leq C \left[\|f\|_{H_{x,y}^1} + \|g\|_{L_{x,y}^2} + \|F\|_{L_t^1 L_{x,y}^2} \right],$$

where $C > 0$ depends on p and might depend on $\text{Vol}(\mathcal{M}^k)$.

Remark 2.3 (Energy estimates). One also has

$$\|u(t)\|_{H_{x,y}^1} + \|\partial_t u(t)\|_{L_{x,y}^2} \leq C \left[\|f\|_{H_{x,y}^1} + \|g\|_{L_{x,y}^2} + \int_0^t \|F(t)\|_{L_{x,y}^2} \right].$$

The proof is divided into four steps:

- (1) We first sketch the proof of Strichartz estimates for the propagator $S_x^1(t)$ given by

$$S_x^1(t)(f, g) = \cos\left(t\sqrt{1 - \Delta_x}\right) f + \frac{\sin\left(t\sqrt{1 - \Delta_x}\right)}{\sqrt{1 - \Delta_x}} g.$$

In that case, we make use of Besov spaces on \mathbf{R}^d . One can find various statements and proofs of Strichartz estimates for (NLKG) in [8, 9, 16, 17, 18, 22, 25, 35, 40, 44] (see for example [17, 16, 18, 25, 40] for proofs).

We will briefly sketch the proof for non-extremal pairs as it is done in [40] with standard TT^* method introduced by Ginibre and Velo for this equation. Extremal pairs are given by Keel and Tao with different methods, using Littlewood-Paley theory in [25]. For more details, we refer the reader to the references given above.

- (2) We use embedding theorems between Sobolev spaces and Lebesgue spaces (classical references here are [1, 56, 57, 58]).

- (3) We use a scaling argument to deduce m -dependent Strichartz estimates for $S_x^{m^2}(t)$ where

$$S_x^{m^2}(t)(f, g) = \cos\left(t\sqrt{m^2 - \Delta_x}\right) f + \frac{\sin\left(t\sqrt{m^2 - \Delta_x}\right)}{\sqrt{m^2 - \Delta_x}} g.$$

- (4) We then can use the Fourier decomposition in the y -variables. Summing on the modes, we obtain Strichartz estimates for $S(t)$ given by (1.5). It is then possible to use embedding theorems for the compact manifold to work in Lebesgue spaces.

Remark 2.4 (The endpoint $p = 2$, $q = 2d/(d - 2)$). In our framework, we only need to handle the endpoint for $d = 4, 5$. For lower d , it is easy to see that $p_0 > 2$. Thus, the results of [25] are enough to deal with the endpoint (see also [29, 30]).

2.3. The results in the Euclidean case. Consider

$$(2.7) \quad u(t, x) = \cos\left(t\sqrt{1 - \Delta_x}\right) f(x) + \frac{\sin\left(t\sqrt{1 - \Delta_x}\right)}{\sqrt{1 - \Delta_x}} g(x) + \int_0^t \frac{\sin\left((t-s)\sqrt{1 - \Delta_x}\right)}{\sqrt{1 - \Delta_x}} F(s) ds.$$

Then we notice that u satisfies

$$\partial_t^2 u - \Delta_x u + u = F, \quad u|_{t=0}(x) = f(x), \quad \partial_t u|_{t=0}(x) = g(x).$$

The statement of the following proposition does not take in account non-sharp pairs and is not as general as in [40] and other references. In fact we just state the estimates in simple spaces that we are going to use.

Proposition 2.5 (Strichartz estimates for $S_x^1(t)$ in Besov Spaces (from [40])). *Let $d \geq 1$ and consider an admissible pair (q, r) , given by Definition 1.2, and s as in (2.4). Consider u given*

by (2.7) for any f, g, F in $\mathcal{S}(\mathbf{R}^d)$. Then

$$(2.8) \quad \|u\|_{L^q(\mathbf{R})B_{r,2}^s(\mathbf{R}^d)} \leq C [\|f\|_{H^1(\mathbf{R}^d)} + \|g\|_{L^2(\mathbf{R}^d)} + \|F\|_{L^1(\mathbf{R})L^2(\mathbf{R}^d)}]$$

where $C > 0$ depends only on the choice of the pair.

In the statement we only put $L_t^1 L_x^2$ since we will not use more general spaces as it will be explained in Remark 2.8.

Sketch of the proof, from [40]. We just sketch the proof for non-extremal pairs, assuming a dispersion inequality. We proceed as in the self-contained proof from [40], with standard TT^* method. We will not recall intermediate results with stationary-non stationary phase methods.

We use the cut-off functions introduced in Section 1.

We denote χ_0 a cut-off function equal to one when ξ is close to zero and χ a cut-off function equal to one on $1/2 < |\xi| < 2$, but supported on $\mathbf{R}^d \setminus \{0\}$. Denote

$$U_0(t) := e^{it\sqrt{1-\Delta_x}} \chi_0(\nabla_x)$$

$$U_\lambda(t) := e^{it\sqrt{1-\Delta_x}} \chi\left(\frac{\nabla_x}{\lambda}\right), \quad \lambda \geq 1,$$

that are bounded in L^2 , uniformly in λ . It is easy to see that for any f, g, F in the Schwartz class, and some fixed q, r the following statements are equivalent

$$(2.9) \quad \|U_\lambda U_\lambda^* F\|_{L_t^q L_x^r} \leq C(\lambda)^2 \|F\|_{L_t^{q'} L_x^{r'}}, \quad \forall F \in \mathcal{S}(\mathbf{R} \times \mathbf{R}^d),$$

$$(2.10) \quad \|U_\lambda^* g\|_{L_x^2} \leq C(\lambda) \|g\|_{L_t^{q'} L_x^{r'}}, \quad \forall g \in \mathcal{S}(\mathbf{R} \times \mathbf{R}^d),$$

$$(2.11) \quad \|U_\lambda f\|_{L_t^q L_x^r} \leq C(\lambda) \|f\|_{L_x^2}, \quad \forall f \in \mathcal{S}(\mathbf{R}^d).$$

Claim 1 : Let $d \geq 1$, $2 < q \leq \infty$, $2 \leq r \leq \infty$ with (q, r) admissible as in Definition 1.2, that is $2/q = d(1/2 - 1/r)$. Then, for any function f in the Schwartz class, one has

$$\|U_\lambda f\|_{L_t^q L_x^r} \leq C(1 + \lambda^2)^\sigma \|f\|_{L_x^2}, \quad \lambda = 0 \text{ or } \lambda \geq 1,$$

$$\text{where } \sigma = \frac{1}{2} \left(\frac{d}{2} + 1 \right) \left(\frac{1}{r'} - \frac{1}{r} \right).$$

Proof of Claim 1: One has

$$\|U_\lambda f\|_{L_x^2} \leq C \|f\|_{L_x^2}.$$

We just get for granted the following dispersion estimate using stationary phase methods and Young's inequality ([8, 44, 17, 16, 18, 40])

$$\|U_\lambda f\|_{L_x^\infty} \leq C_0 \lambda^{1+d/2} t^{-d/2} \|f\|_{L_x^1}.$$

Interpolating, one gets

$$\|U_\lambda f\|_{L_x^r} \leq C_1 \left(\lambda^{1+d/2} t^{-d/2} \right)^{\frac{1}{r'} - \frac{1}{r}} \|f\|_{L_x^{r'}}.$$

As a consequence

$$\|U_\lambda U_\lambda^* F(t)\|_{L_x^r} \leq C \int_{-\infty}^{+\infty} \left(\lambda^{1+d/2} (t-s)^{-d/2} \right)^{\frac{1}{r'} - \frac{1}{r}} \|F(s)\|_{L_{x'}^{r'}} ds$$

and Hardy-Littlewood-Sobolev inequality writes

$$\left\| f * \frac{1}{|t|^\alpha} \right\|_{L_t^s} \leq C(\rho, s, \alpha) \|f\|_{L_t^\rho}, \quad 1 + 1/s = \alpha + 1/\rho, \quad \alpha \in (0, 1).$$

Using this inequality, one gets

$$\left(\int_{\mathbf{R}} \|U_\lambda U_\lambda^* F(t)\|_{L_x^r}^q \right)^{1/q} \leq C' \lambda^{(\frac{d}{2}+1)(\frac{1}{r'} - \frac{1}{r})} \left(\int_{\mathbf{R}} \left| \int_{-\infty}^{+\infty} \frac{1}{(t-s)^{d/2(\frac{1}{r'} - \frac{1}{r})}} \|F(s)\|_{L_{x'}^{r'}} ds \right|^q dt \right)^{1/q},$$

$$\|U_\lambda U_\lambda^* F(t)\|_{L_t^q L_x^r} \leq C' \lambda^{(\frac{d}{2}+1)(\frac{1}{r'} - \frac{1}{r})} \|F\|_{L_t^\rho L_{x'}^{r'}},$$

with $\alpha = d/2(\frac{1}{r'} - \frac{1}{r})$, and $\rho = q'$. Noticing that for $\lambda \geq 1$, one can bound λ by $\sqrt{1 + \lambda^2}$, we get (2.9) for admissible pairs, except the endpoint $q = 2$, with

$$C(\lambda)^2 = C' (1 + \lambda^2)^{\frac{1}{2}(1+\frac{d}{2})(\frac{1}{r'} - \frac{1}{r})}.$$

Now that (2.9) is proved, one also has (2.10) and (2.11) for admissible pairs with $C(\lambda)$. ■

Let us recall that Besov and Sobolev norms can be expressed with the partition of the unity given in the introduction, as

$$\begin{aligned} \|f\|_{B_{r,2}^\sigma} &:= \|P_0 f\|_{L^r} + \left(\sum_{j>0} 2^{2\sigma j} \|P_j f\|_{L^r}^2 \right)^{1/2} \\ &\simeq \|2^{\sigma j} \|P_j f\|_{L^r}\|_{l_j^2}, \\ \|f\|_{H^\sigma} &:= \|P_0 f\|_{L^2} + \left(\sum_{j>0} 2^{2\sigma j} \|P_j f\|_{L^2}^2 \right)^{1/2}. \end{aligned}$$

Then, taking $\lambda \in \{0\} \cup \{2^j, j \in \mathbf{N}\}$ and summing in square the estimates from **Claim 1**, one obtains

$$\|e^{it\sqrt{1-\Delta}} f\|_{L_t^q B_{r,2}^0} \leq C \|f\|_{H^\sigma},$$

and so

$$\|e^{it\sqrt{1-\Delta}} f\|_{L_t^q B_{r,2}^{1-\sigma}} \leq C \|f\|_{H^1}.$$

Consider u given by (2.7).

Claim 2: Any solution of

$$\partial_t^2 u - \Delta_x u + u = F, \quad u(0, x) = f(x), \quad \partial_t u(0, x) = g(x),$$

in $\mathbf{R} \times \mathbf{R}^d$ satisfies the estimates

$$\|u\|_{L_t^q B_{r,2}^s} \leq C \left[\|f\|_{H^1} + \|g\|_{L^2} + \|F\|_{L_t^1 L_x^2} \right],$$

where (q, r) are admissible and s is as in (2.4).

Proof of Claim 2: The hompogeneous case $F = 0$ is given before. Consider the case $F \neq 0$, but $f = g = 0$. The whole estimate is obtained combining both cases.

For any $j \geq 0$, one wants to prove

$$\|P_j u\|_{L_t^q B_{r,2}^s} \leq C \|F\|_{L_t^1 L_x^2}.$$

Noticing that for $C(\lambda, r) \simeq \lambda^{\frac{1}{2}(\frac{d}{2}+1)(\frac{1}{r^*}-\frac{1}{r})}$

$$\begin{aligned} U_0 : L^2 &\rightarrow L^q L^r, & U_\lambda : C(\lambda, r) L^2 &\rightarrow L^q L^r, & U_0 U_0^* : L^1 L^2 &\rightarrow L^q L^r, \\ U_0^* : L^1 L^2 &\rightarrow L^2, & U_\lambda^* : L^1 L^2 &\rightarrow L^2, & U_\lambda U_\lambda^* : C(\lambda, r) L^1 L^2 &\rightarrow L^q L^r, \end{aligned}$$

and writing

$$U_0 U_0^* F(t) = \int_{\mathbf{R}} e^{i(t-s)\sqrt{1-\Delta}} \chi_0(\xi)^2 F(s) ds,$$

we use Christ-Kiselev Lemma which allows to handle the integral on $[0, t]$, and the same argument can be used for the $U_\lambda U_\lambda^*$ term, providing the good weight in λ . The estimates are obtained proceeding as before. ■

Finally, noticing that the endpoint $q = 2$ is not given with the previous method, but is given in [25] and interpolating the associated estimates with the energy estimates, one obtains the more general Strichartz estimates stated in the Proposition, by interpolation. □

2.4. Statements of embedding theorems. In this section, we state the embedding theorems without proofs and apply them to the estimates given by (2.8). From [57, 58] one states

Theorem 2.6 (Embedding theorems). *Let $d \geq 1$, $s > 0$, $1 \leq r, \rho \leq \infty$ and consider Besov $B_{r,2}^s(\mathbf{R}^d)$ and Lebesgue $L^\rho(\mathbf{R}^d)$ spaces*

- (1) $B_{r,2}^s(\mathbf{R}^d) \subset L^r(\mathbf{R}^d)$,
- (2) $B_{r,2}^s(\mathbf{R}^d) \subset L^{r^*}(\mathbf{R}^d)$ if $r^* \geq 2$, where $s - d/r = -d/r^*$.

Interpolation gives the Lebesgue spaces with exponents ρ lying between r and r^ , for $r \geq 2$*

- (3) $B_{r,2}^s(\mathbf{R}^d) \subset L^\rho(\mathbf{R}^d)$ if $2 \leq r \leq \rho \leq r^*$, where $s - d/r = -d/r^*$.

Point 1 of Theorem 2.6 can be found in [57](Section 2.3.2, Estimate (24) in Remark 3, p.97). Point 2 can be found in [58] (Section 1.9.1, Theorem 1.73, Estimate (1.200), p.40) and can be deduced from a more general embedding involving Triebel-Lizorkin spaces somehow linked with Sobolev spaces. Such results can also be deduced from [1, 56] and references therein. One can easily see that the relation between r, r^* and ρ are the same as for usual Sobolev embeddings which is quite natural since Besov spaces are obtained with interpolation between Sobolev spaces. Note that this theorem gives the Lebesgue spaces used in [40] for the cubic NLKG on \mathbf{R} (see Exercise 2.45, p. 75) and \mathbf{R}^3 (see Exercise 2.42, p. 74 and Lemma 2.46 p.77).

We deduce

Proposition 2.7. *Let $d \geq 1$. Consider an admissible pair (q, r) given by Definition 1.2, and s as in (2.4). Consider u given by (2.7). Then for any ρ such that $2 \leq r \leq \rho \leq r^*$, where $s - d/r = -d/r^*$, we have*

$$(2.12) \quad \|u\|_{L_t^q L_x^\rho} \leq C \left[\|f\|_{H_x^1} + \|g\|_{L_x^2} + \|F\|_{L_t^1 L_x^2} \right],$$

The proof is immediate, applying Theorem 2.6 to (2.8).

Remark 2.8. Strichartz estimates proved in [40] are more general than Proposition 2.2, since they allow the source term to be in some $L^{q'} B_{r', 2}^{(1-s')}(R^d)$, where (q, r) admissible, s' as in (2.4). However, it is well known that embeddings of type

$$W^{\sigma, r}(R^d) \subset B_{r, 2}^\sigma(R^d), \quad 1 < r \leq 2,$$

are valid and the only way to obtain a Lebesgue space is to consider $\sigma = 0$. Thus, the only (q', r') giving Lebesgue exponents for the source term is $(1, 2)$ with $s' = 1$.

2.5. The scaling argument for $m^2 \neq 1$. All the estimates enunciated before are proved for $m^2 = 1$. As explained in Section 2.1, they need to be adapted to different masses, which is performed using a scaling argument.

Consider $\lambda > 0$ and

$$(2.13) \quad u_\lambda(t, x) = \cos\left(t\sqrt{\lambda - \Delta_x}\right) f_\lambda(x) + \frac{\sin\left(t\sqrt{\lambda - \Delta_x}\right)}{\sqrt{\lambda - \Delta_x}} g_\lambda(x) + \int_0^t \frac{\sin\left((t-s)\sqrt{\lambda - \Delta_x}\right)}{\sqrt{\lambda - \Delta_x}} F_\lambda(s) ds.$$

One sees that $u_\lambda(t, x) = u(\sqrt{\lambda}t, \sqrt{\lambda}x)$ where u is given by (2.7) and that it satisfies

$$(2.14) \quad \partial_t^2 u_\lambda - \Delta_x u_\lambda + \lambda u_\lambda = F_\lambda, \quad u_\lambda|_{t=0}(x) = f_\lambda(x), \quad \partial_t u_\lambda|_{t=0}(x) = g_\lambda(x),$$

where

$$f_\lambda(x) = f(\sqrt{\lambda}x), \quad g_\lambda(x) = \sqrt{\lambda}g(\sqrt{\lambda}x), \quad F_\lambda(t, x) = \lambda F(\sqrt{\lambda}t, \sqrt{\lambda}x).$$

Then

Proposition 2.9. *Let $d \geq 1$ and consider some (q, r) admissible as in Definition 1.2, s given by (2.4) and ρ such that $2 \leq r \leq \rho \leq r^*$, where $s - d/r = -d/r^*$. Consider u given by (2.7) for which (2.12) holds. Then for u_λ given by (2.13), one has*

$$(2.15) \quad \lambda^{\frac{1}{2}(\frac{d}{\rho} + \frac{1}{q} + 1 - \frac{d}{2})} \|u_\lambda\|_{L_t^q L_x^\rho} \leq C \left[\sqrt{\lambda} \|f_\lambda\|_{L_x^2} + \|f_\lambda\|_{\dot{H}_x^1} + \|g_\lambda\|_{L_x^2} + \|F_\lambda\|_{L_t^1 L_x^2} \right].$$

Proof. Considering u and u_λ , respectively given by (2.7) and (2.13), an easy computation shows that

$$\begin{aligned} \star \quad & \|u_\lambda\|_{L^q L^\rho} = \lambda^{\frac{-1}{2}(\frac{d}{\rho} + \frac{1}{q})} \|u\|_{L^q L^\rho}, \\ \star \quad & \|f_\lambda\|_{H^1} = \|f_\lambda\|_{L^2} + \|f_\lambda\|_{\dot{H}^1} = \lambda^{\frac{-d}{4}} \|f\|_{L^2} + \lambda^{\frac{1}{2} - \frac{d}{4}} \|f\|_{\dot{H}^1}, \\ \star \quad & \|g_\lambda\|_{L^2} = \lambda^{\frac{1}{2} - \frac{d}{4}} \|g\|_{L^2}, \end{aligned}$$

$$\star \|F_\lambda\|_{L^1 L^2} = \lambda^{\frac{1}{2} - \frac{d}{4}} \|F\|_{L^1 L^2},$$

and combining those estimates, one gets (2.15). \square

2.6. End of the proof of Proposition 2.2.

2.6.1. *General computations.* The next step is to consider (2.3) taking $\lambda = 1 + \lambda_j$ with λ_j given in (1.4) and follow the strategy described in Section 2.1.

We recall the system of equations (2.3) after a decomposition on (1.4)

$$\begin{aligned} u(t, x, y) &= \sum_j u_j(t, x) \Phi_j(y) \\ F(t, x, y) &= \sum_j F_j(t, x) \Phi_j(y) \\ f(x, y) &= \sum_j f_j(x) \Phi_j(y) \\ g(x, y) &= \sum_j g_j(x) \Phi_j(y), \end{aligned}$$

that is

$$\partial_t^2 u_j - \Delta_x u_j + u_j + \lambda_j u_j = F_j, \quad u_j|_{t=0}(x) = f_j(x), \quad \partial_t u_j|_{t=0}(x) = g_j(x).$$

From (2.15), with $\lambda = 1 + \lambda_j$, one has for an appropriate choice of pairs that will be given later

$$(\lambda_j + 1)^{\frac{1}{2}(\frac{d}{\rho} + \frac{1}{q} + 1 - \frac{d}{2})} \|u_j\|_{L_t^q L_x^\rho} \leq C \left[(\lambda_j + 1)^{1/2} \|f_j\|_{L_x^2} + \|f_j\|_{\dot{H}_x^1} + \|g_j\|_{L_x^2} + \|F_j\|_{L_t^1 L_x^2} \right].$$

Then, summing in j the square as in [60] one obtains

$$\left\| (\lambda_j + 1)^{\frac{1}{2}(\frac{d}{\rho} + \frac{1}{q} + 1 - \frac{d}{2})} u_j \right\|_{l_j^2 L_t^q L_x^\rho} \leq C \left[\left\| (\lambda_j + 1)^{1/2} f_j \right\|_{l_j^2 L_x^2} + \|f_j\|_{l_j^2 \dot{H}_x^1} + \|g_j\|_{l_j^2 L_x^2} + \|F_j\|_{l_j^2 L_t^1 L_x^2} \right].$$

Minkowski inequality for the left handside and for the source term (since $\max(1, 2) \leq 2 \leq \min(q, \rho)$) gives

$$\left\| (\lambda_j + 1)^{\frac{1}{2}(\frac{d}{\rho} + \frac{1}{q} + 1 - \frac{d}{2})} u_j \right\|_{L_t^q L_x^\rho l_j^2} \leq C \left[\left\| (\lambda_j + 1)^{1/2} f_j \right\|_{L_x^2 l_j^2} + \|f_j\|_{\dot{H}_x^1 l_j^2} + \|g_j\|_{L_x^2 l_j^2} + \|F_j\|_{L_t^1 L_x^2 l_j^2} \right].$$

Using Plancherel identity we notice that

$$\begin{aligned} & \underbrace{\|\sqrt{1 + \lambda_j} f_j\|_{l_j^2 L_x^2}}_{\simeq \|f\|_{L_{x,y}^2} + \|\lambda_j|^{1/2} f\|_{L_{x,y}^2}} + \underbrace{\|f_j\|_{l_j^2 \dot{H}_x^1}}_{\simeq \|\partial_x f\|_{L_{x,y}^2}} \simeq \|f\|_{H_{x,y}^1}. \end{aligned}$$

Then thanks to the decomposition on (1.4), one is able to handle the y -variable to obtain

$$\begin{aligned} \left\| (1 - \Delta_y)^{\frac{1}{2}(\frac{d}{\rho} + \frac{1}{q} + 1 - \frac{d}{2})} u \right\|_{L_t^q L_x^\rho L_y^2} &\leq C \left[\|f\|_{H_{x,y}^1} + \|g\|_{L_{x,y}^2} + \|F\|_{L_t^1 L_x^2 L_y^2} \right] \\ \|u\|_{L_t^q L_x^\rho H_y^\gamma} &\leq C \left[\|f\|_{H_{x,y}^1} + \|g\|_{L_{x,y}^2} + \|F\|_{L_t^1 L_x^2 L_y^2} \right], \end{aligned}$$

where

$$\gamma = \left(\frac{d}{\rho} + \frac{1}{q} + 1 - \frac{d}{2} \right) \geq 0, \quad \text{since } 2 \leq r \leq \rho \leq r^*.$$

2.6.2. *Restrictions to prove (2.5) and (2.6): ideas and remarks.* Let us now focus on the exponents in Proposition 2.2. We make the following requirements on the exponents:

- (i): We fix $q = p$, then r is uniquely defined. We then find all possible p for which $\rho = 2p$.
- (ii): p should also satisfy $H^\gamma(\mathcal{M}^k) \subset L^{2p}(\mathcal{M}^k)$.

We first notice that for $(q, \rho) = (p, 2p)$, $H^\gamma(\mathcal{M}^k) \subset L^2(\mathcal{M}^k)$ where γ is given by (2.19). Let us now make the following remark on (ii): we introduce

$$p_{sob} = \frac{d+2}{d+k-2}$$

and notice that for every $k \geq 1$, $p_{sob} < p_c$. The Sobolev embedding

$$H^\gamma(\mathcal{M}^k) \subset L^{2p}(\mathcal{M}^k) \text{ if } k \geq 2\gamma > 0 \text{ and } \frac{k}{k-2\gamma} \geq 2p,$$

gives $p \geq p_{sob}$. But if $k < 2\gamma$, Morrey estimates are available and

$$H^\gamma(\mathcal{M}^k) \subset L^\infty(\mathcal{M}^k).$$

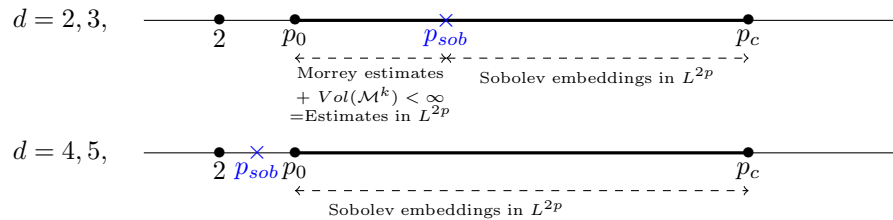
$Vol(\mathcal{M}^k)$ is finite,

$$(2.16) \quad \|f\|_{L_y^q} \leq \|f\|_{L_y^\infty} \times Vol(\mathcal{M}^k)^{1/q}, \quad \forall q \geq 1,$$

which is true for the particular case $q = 2p$. Let us notice that since $p \geq 2$ and $H^\gamma(\mathcal{M}^k) \subset L^2(\mathcal{M}^k)$ then $H^\gamma(\mathcal{M}^k) \subset L^{2p}(\mathcal{M}^k)$ even for a manifold with infinite volume. Hence

$$H^\gamma(\mathcal{M}^k) \subset L^{2p}(\mathcal{M}^k) \text{ if } k < 2\gamma \text{ or } k \geq 2\gamma, \text{ and } \frac{2k}{k-2\gamma} \geq 2p.$$

As an example, consider $\mathbf{R}^3 \times \mathcal{M}^1$. Point (1) of Theorem 1.3 gives energy scattering for $5/2 \leq p \leq 3$ but the interval $[7/3, 5/2)$ is not covered. However we can deal with global existence and energy scattering when $7/3 \leq p < 5/2$ using (2.16). The only cases that need to handle two different regions for p are $\mathbf{R}^d \times \mathcal{M}^1$ for $d = 2, 3$. In fact, in our framework, the only case allowed with $d = 1$ is $\mathbf{R} \times \mathcal{M}^2$ handled with Point (i) of Theorem 1.3. And for $d = 4, 5$, it is easy to see that $p_{sob} \geq p_0$:



2.6.3. *Claims and proofs.* Set $q = p$, then

$$(2.17) \quad r = \frac{2dp}{dp-4}$$

$$(2.18) \quad s = \frac{dp-d-2}{dp}$$

$$(2.19) \quad \gamma = \frac{d+2+2p-dp}{2p},$$

and we notice that if $d = 1$, we should restrict to $p \geq 4$. We recall that the exponent p in our framework satisfies $p_0 \leq p \leq p_c$ that is

- ★ $d = 1, p \geq 5$,
- ★ $d = 2, 3 \leq p \leq 1 + 4/k$,
- ★ $d \geq 3, \max(2, 1 + 4/d) \leq p \leq 1 + 4/(d + k - 2)$,

and notice that to apply Theorem 2.6 we need $s > 0$ and $r^* = dr/(d - sr) \geq r \geq 2$. The condition $s > 0$ is fulfilled in the following cases

- ★ $d = 1, p > 3$,
- ★ $d = 2$, for all $p > 2$,
- ★ $d \geq 3$, for all $p > 1 + 2/d$,

which is always fulfilled for $p \geq p_0$.

Claims: Consider $1 \leq d \leq 5$ and $p_0 \leq p$, given by (1.3).

a. Then for any (p, r) admissible as in (2.17) and s given by (2.18), one has

$$B_{r,2}^s(\mathbf{R}^d) \subset L^{2p}(\mathbf{R}^d), \quad \text{if } p \leq \frac{d^2 + 2d - 4}{d^2 - 2d} \text{ when } d \geq 3.$$

b. For any couple $(p, 2p)$ satisfying Claim a., and for any $k = 1, 2, k = 2$ if $d = 1$,

$$H^\gamma(\mathcal{M}^k) \subset L^{2p}(\mathcal{M}^k) \quad \text{if } p \leq p_c.$$

where γ is given by (2.19).

Proof:

$d = 1$: From [40], one gets for any $4 \leq p \leq \infty, 2 \leq r \leq \infty$, given by (2.17) and $s = (p - 3)/p$,

$$\|u_j\|_{L_t^q L_x^p} \lesssim \|f_j\|_{H_x^1} + \|g_j\|_{L_x^2} + \|F_j\|_{L_t^1 L_x^2},$$

provided that

$$\rho \geq r \text{ and } \rho \text{ is such that } s \geq \frac{1}{r} - \frac{1}{\rho}.$$

Replacing ρ with $2p$, $r \leq 2p$ yields $p \geq p_0$ and for any $p \geq p_0$, we have

$$\frac{p-3}{p} \geq \frac{1}{r} - \frac{1}{2p} = \frac{p-5}{2p},$$

which gives Claim a.

We compute γ for the couple $(q, \rho) = (p, 2p)$ which gives $\gamma = (3 + p)/2p$. It is then easy to see that for $k = 2$

$$H^\gamma(\mathcal{M}^k) \subset L^{2p}(\mathcal{M}^k) \text{ if } \frac{2k}{k - 2\gamma} \geq 2p \text{ that is } p = p_c = 5.$$

which gives Claim b and c for $d = 1$.

$d = 2$: We compute

$$sr = \frac{2(2p-4)}{2p-4} = 2.$$

Thus, we have $B_{r,2}^s(\mathbf{R}^2) \subset L^\rho(\mathbf{R}^2)$, for all $r \leq \rho < r^* = \infty$. So it is enough to check $2p \geq r$ which is true for any $p \geq p_0$ and Claim a is proved.

We then see $\gamma = 2/p$ and

$$H^{2/p}(\mathcal{M}^k) \subset L^{2p}(\mathcal{M}^k) \text{ if } k < \frac{4}{p} \text{ or } k \geq \frac{4}{p} \text{ and } \frac{2k}{k - 4/p} \geq 2p,$$

that is $p \leq p_c$.

$d \geq 3$: The restrictions $2 \leq r \leq 2p \leq r^* = dr/(d - sr)$ gives

$$1 + \frac{4}{d} \leq p \leq \frac{d^2 + 2d - 4}{d^2 - 2d},$$

and since we only deal with $p \geq 2$, it is easy to check that the condition

$$\frac{d^2 + 2d - 4}{d^2 - 2d} \geq 2$$

allows us to deal with $d \leq 5$, which gives Claim a for $d \geq 3$. We then see that

$$H^\gamma(\mathcal{M}^k) \subset L^2(\mathcal{M}^k) \text{ if } p \geq p_0$$

and

$$H^\gamma(\mathcal{M}^k) \subset L^{2p}(\mathcal{M}^k) \text{ if } k < 2\gamma \text{ or } k \geq 2\gamma \text{ and } \frac{2k}{k - 2\gamma} \geq 2p,$$

gives the condition

$$p \leq \frac{d + k + 2}{d + k - 2} = p_c.$$

We finally notice that $p_c \leq \frac{d^2 + 2d - 4}{d^2 - 2d}$ in the framework of Theorem 1.3.

To obtain (2.6) in Proposition 2.2, we restrict to the cases covered by Claim a and b, whereas for (2.5), restrictions carried by Claim a and c will give the results.

We sum up the restrictions in the following table

Conditions	$d = 1$	$d = 2$	$d \geq 3$
$(q, \rho) = (p, 2p)$	$p_0 \leq p$		$p_0 \leq p \leq \frac{d^2 + 2d - 4}{d^2 - 2d}$
Possible p to have (2.6)	$k \geq 2,$ $p \geq 5,$	$k \geq 1,$ $p \geq 3$	$k \geq 1, d \leq 5,$ $p_0 \leq p \leq \frac{d^2 + 2d - 4}{d^2 - 2d}$
$H^\gamma(\mathcal{M}^k) \subset L^{2p}(\mathcal{M}^k)$	$p_0 \leq p \leq p_c$		
Possible p to have (2.5)	$k = 2,$ $p = 5$	$k = 1, 2,$ $3 \leq p \leq 1 + \frac{4}{k}$	$k = 1, 2, d \leq 5, d + k \leq 6$ $p_0 \leq p \leq p_c$

which concludes the proof of Proposition 2.2.

3. PROOF OF THEOREM 1.3

3.1. Global existence of the solution. Let us first remark that (at least local) existence theory is available on smooth (C^∞) Riemannian manifolds without boundaries, of dimension larger than 3 (in view of [23, 24]), for defocusing and energy critical nonlinearities. The finite speed propagation is a key point in the analysis. We will not use those results here, since we also deal with focusing energy (sub-)critical cases and will perform a classical fixed point argument in the small data framework. For simplicity, we write $L_t^p L_{x,y}^{2p} = L^p(\mathbf{R}, L_{x,y}^{2p})$. We recall that we only consider $p \geq 2$.

$$\begin{aligned} \Phi_0 u(t) = \cos\left(t\sqrt{1-\Delta_{x,y}}\right) f(x) + \frac{\sin\left(t\sqrt{1-\Delta_{x,y}}\right)}{\sqrt{1-\Delta_{x,y}}} g(x) + \\ \int_0^t \frac{\sin\left((t-s)\sqrt{1-\Delta_{x,y}}\right)}{\sqrt{1-\Delta_{x,y}}} (|u|^{p-1}u)(s) ds. \end{aligned}$$

Then (1.1) is equivalent to $\Phi_0 u(t) = u(t)$. Applying (2.5) with $F = |u|^{p-1}u$ on $\Phi_0 u(t)$ we get

$$\begin{aligned} \|\Phi_0 u(t)\|_{L_t^p L_{x,y}^{2p}} &\leq C \left[\|f\|_{H_{x,y}^1} + \|g\|_{L_{x,y}^2} + \| |u|^{p-1}u \|_{L_t^1 L_{x,y}^2} \right] \\ &\leq C \left[\|f\|_{H_{x,y}^1} + \|g\|_{L_{x,y}^2} + \|u\|_{L_t^p L_{x,y}^{2p}}^p \right]. \end{aligned}$$

We define for $R > 0$

$$X_R = \left\{ \varphi \mid \|\varphi\|_{L_t^p L_{x,y}^{2p}} \leq R \right\},$$

and $\delta > 0$ such that $\|f\|_{H_{x,y}^1} + \|g\|_{L_{x,y}^2} < \delta$, and will specify δ later. Taking $R = 2C\delta$, any $u \in X_{2C\delta}$ satisfies

$$\|\Phi_0 u(t)\|_{L_t^p L_{x,y}^{2p}} \leq C \left[\|f\|_{H_{x,y}^1} + \|g\|_{L_{x,y}^2} + \|u\|_{L_t^p L_{x,y}^{2p}}^p \right] \leq 2C\delta \left(\frac{1}{2} + (2\delta)^{p-1} C^p \right).$$

Taking $\delta \leq \frac{1}{(2C)^{p/(p-1)}}$ one obtains that Φ_0 maps $X_{2C\delta}$ into itself.

Consider two solution $v, w \in X_{2C\delta}$ with the same initial data. Then with Proposition 2.2, we have

$$\|\Phi_0 v - \Phi_0 w\|_{L_t^p L_{x,y}^{2p}} \leq C \| |v|^{p-1}v - |w|^{p-1}w \|_{L_t^1 L_{x,y}^2}.$$

We can proceed as in [10] to handle non integer nonlinearities.

$$||v|^{p-1}v - |w|^{p-1}w| \leq \tilde{C} (|v|^{p-1} + |w|^{p-1}) |v - w|.$$

$$\|\Phi_0 v - \Phi_0 w\|_{L_t^p L_{x,y}^{2p}} \leq \tilde{C} \|v - w\|_{L_t^p L_{x,y}^{2p}} \left[\|v\|_{L_t^p L_{x,y}^{2p}}^{p-1} + \|w\|_{L_t^p L_{x,y}^{2p}}^{p-1} \right].$$

Any v, w in $X_{2C\delta}$ satisfy

$$\|\Phi_0 v - \Phi_0 w\|_{L_t^p L_{x,y}^{2p}} \leq \left(2^p \delta^{p-1} \max(C, \tilde{C})^p \right) \|v - w\|_{L_t^p L_{x,y}^{2p}}.$$

Hence choosing δ such that $2^p \delta^{p-1} \max(C, \tilde{C})^p < 1$, Φ_0 is a contraction on $X_{2C\delta}$.

Finally, one can deduce global existence of the (unique) solution in $L^p(\mathbf{R}, L_{x,y}^{2p})$ for any $(f, g) \in H_{x,y}^1 \times L_{x,y}^2$ such that

$$\|f\|_{H_{x,y}^1} + \|g\|_{L_{x,y}^2} < \delta_0 = \min \left(\frac{1}{(2C)^{p/(p-1)}}, \frac{1}{\left(2 \max(C, \tilde{C})\right)^{p/(p-1)}} \right).$$

Then, using the energy estimate (2.3) and $u \in L^p(\mathbf{R}, L_{x,y}^{2p})$, one gets a solution u such that

$$u(t, x, y) \in C^0(\mathbf{R}, H_{x,y}^1) \cap C^1(\mathbf{R}, L_{x,y}^2), \quad \text{and so} \quad \partial_t u(t, x, y) \in C^0(\mathbf{R}, L_{x,y}^2).$$

3.2. Scattering results. Scattering is proved in the fashion of [40]. We write

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\Delta_{x,y} + 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} + \begin{pmatrix} 0 \\ \pm |u|^{p-1} u \end{pmatrix},$$

we get with $e^{tH} = \begin{pmatrix} \cos(t\sqrt{1-\Delta_{x,y}}) & \frac{\sin(t\sqrt{1-\Delta_{x,y}})}{\sqrt{1-\Delta_{x,y}}} \\ -\sin(t\sqrt{1-\Delta_{x,y}}) \cdot (\sqrt{1-\Delta_{x,y}}) & \cos(t\sqrt{1-\Delta_{x,y}}) \end{pmatrix},$

which is unitary on the energy space $H_{x,y}^1 \times L_{x,y}^2$

$$\begin{aligned} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} &= e^{tH} \begin{pmatrix} f \\ g \end{pmatrix} + \int_0^t e^{(t-s)H} \begin{pmatrix} 0 \\ \pm |u|^{p-1} u \end{pmatrix} ds \\ e^{-tH} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} &= \begin{pmatrix} f \\ g \end{pmatrix} + \int_0^t e^{(-s)H} \begin{pmatrix} 0 \\ \pm |u|^{p-1} u \end{pmatrix} ds. \end{aligned}$$

Writing $V(t) = e^{-tH} \begin{pmatrix} u \\ \partial_t u \end{pmatrix}$, and considering $0 < \tau < t$ it is easy to check that

$$\|V(t) - V(\tau)\|_{H^1 \times L^2} \leq C \int_\tau^t \| |u|^{p-1} u(s) \|_{L^2} ds \leq C \|u\|_{L^p([\tau, t], L^{2p})}^p,$$

where the latter norm tends to zero when t, τ tends to $\pm\infty$ since the solution belongs to $L^p(\mathbf{R}, L_{x,y}^{2p})$.

Therefore, there exist $(f^\pm, g^\pm) \in H_{x,y}^1 \times L_{x,y}^2$ such that $V(t) \rightarrow \begin{pmatrix} f^\pm \\ g^\pm \end{pmatrix}$ in $H^1 \times L^2$ as $t \rightarrow \pm\infty$.

4. PROOF OF THEOREM 1.4

As it is noticed in [60, 54], $(1 - \Delta_y)^{\gamma/2}$ commutes with the linear part of (1.1) on $\mathbf{R}^d \times \mathcal{M}^k$ for any $\gamma \geq 0$. Let us consider the assumptions for which Proposition 2.2 holds. From point (2) of

Proposition 2.2, we have

$$\begin{aligned}
(4.1) \quad & \left\| \cos \left(t \cdot \sqrt{1 - \Delta_{x,y}} \right) (1 - \Delta_y)^{\gamma/2} f \right\|_{L_t^p L_x^{2p} L_y^2} \\
& + \left\| \frac{\sin \left(t \cdot \sqrt{1 - \Delta_{x,y}} \right)}{\sqrt{1 - \Delta_{x,y}}} (1 - \Delta_y)^{\gamma/2} g \right\|_{L_t^p L_x^{2p} L_y^2} \\
& + \left\| \int_0^t \frac{\sin \left((t-s) \cdot \sqrt{1 - \Delta_{x,y}} \right)}{\sqrt{1 - \Delta_{x,y}}} (1 - \Delta_y)^{\gamma/2} F(s) ds \right\|_{L_t^p L_x^{2p} L_y^2} \\
& \leq C \left[\|(1 - \Delta_y)^{\gamma/2} f\|_{H_{x,y}^1} + \|(1 - \Delta_y)^{\gamma/2} g\|_{L_{x,y}^2} + \|(1 - \Delta_y)^{\gamma/2} F\|_{L_t^1 L_{x,y}^2} \right],
\end{aligned}$$

where $C > 0$ depends on the choice of the pairs. The key point is that for $\gamma > k/2$, $H^\gamma(\mathcal{M}_y^k)$ is an algebra. Therefore, considering a product source term $F(t, x, y) = \prod_{i=1}^I u_i(t, x, y)$, one can write for $I \in \mathbb{N}$

$$\left\| \prod_{i=1}^I u_i \right\|_{H_y^\gamma} \leq C \prod_{i=1}^I \|u_i\|_{H_y^\gamma}.$$

Hence for $u_i \in \{u, \bar{u}\}$, $I = p$

$$\left\| \prod_{i=1}^p \|u_i\|_{H_y^\gamma} \right\|_{L_t^1 L_{x,y}^2} \leq C \left\| \|u\|_{H_y^\gamma}^p \right\|_{L_t^1 L_x^2} \leq C \|u\|_{L_t^p L_x^{2p} H_y^\gamma}^p.$$

The rest of the proof of global existence follows easily by using same arguments as for the proof of Theorem 1.3 with $(1 - \Delta_y)^{\gamma/2} u$. Note that the energy estimate together with

$$\|u\|_{L_t^p L_{x,y}^{2p}}^p \lesssim \|u\|_{L_t^p L_x^{2p} L_y^\infty}^p \lesssim \|u\|_{L_t^p L_x^{2p} H_y^\gamma}^p$$

gives the continuity in time of the solution.

With the estimate, we see that writing $V(t) = e^{-tH} \begin{pmatrix} u \\ \partial_t u \end{pmatrix}$, and considering $0 < \tau < t$ it is easy to check that

$$\begin{aligned}
\|V(t) - V(\tau)\|_{(H_{x,y}^{1,\gamma} \times H_{x,y}^{0,\gamma}) \cap (H_{x,y}^1 \times L_{x,y}^2)} & \leq C \int_\tau^t \| |u|^{p-1} u(s) \|_{L_{x,y}^2} ds \\
& \leq C \|u\|_{L^p([\tau,t], L_{x,y}^{2p})}^p \leq C \|u\|_{L^p([\tau,t], L_x^{2p} H_y^\gamma)}^p,
\end{aligned}$$

where the latter norm tends to zero when t, τ tends to $\pm\infty$ since the solution belongs to $L^p(\mathbf{R}, L_x^{2p} H_y^\gamma)$, $\gamma > k/2$. Therefore, there exist $(f^\pm, g^\pm) \in (H_{x,y}^{1,\gamma} \times H_{x,y}^{0,\gamma}) \cap (H_{x,y}^1 \times L_{x,y}^2)$ such that $V(t) \rightarrow \begin{pmatrix} f^\pm \\ g^\pm \end{pmatrix}$ in $(H_{x,y}^{1,\gamma} \times H_{x,y}^{0,\gamma}) \cap (H_{x,y}^1 \times L_{x,y}^2)$ as $t \rightarrow \pm\infty$.

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(L. HARI AND N. VISCIGLIA)
 DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PISA,
 LARGO BRUNO PONTECORVO 5,
 56127 PISA, ITALIA.

E-mail address: lhary@mail.dm.unipi.it,

E-mail address: viscigli@dm.unipi.it